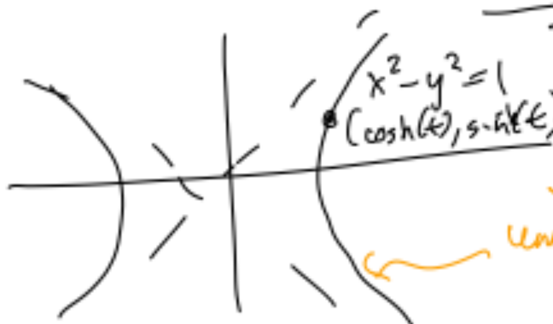
Example calculation of surface area.

<u>Interlude:</u> hyperbolic trig fcns.

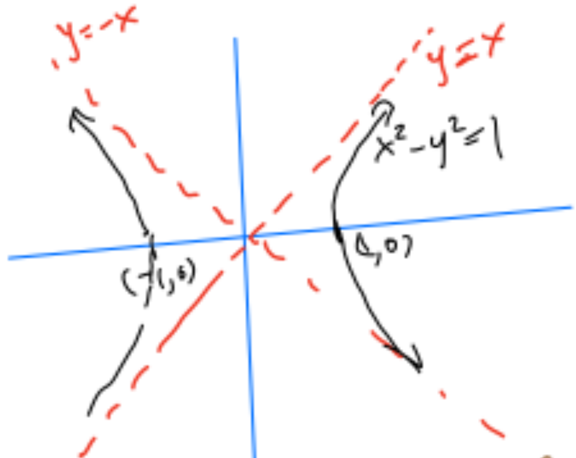
$$\left.\begin{aligned}\cosh(x) &= \frac{e^x+e^{-x}}{2}\\ \sinh(x) &= \frac{e^x-e^{-x}}{2}.\end{aligned}\right)\ \text{hyperbolic trig fcn.}$$



$(\cosh(t), \sinh(t))$ is a pt on $x^2-y^2=1$

unit hyperbola.

Why does this work?



$$\cosh^2(t)-\sinh^2(t) = \left(\frac{e^t+e^{-t}}{2}\right)^2 - \left(\frac{e^t-e^{-t}}{2}\right)^2$$

$$= \frac{e^{2t}+e^{-2t}+2e^te^{-t}}{4} - \frac{e^{2t}+e^{-2t}-2e^te^{-t}}{4}$$

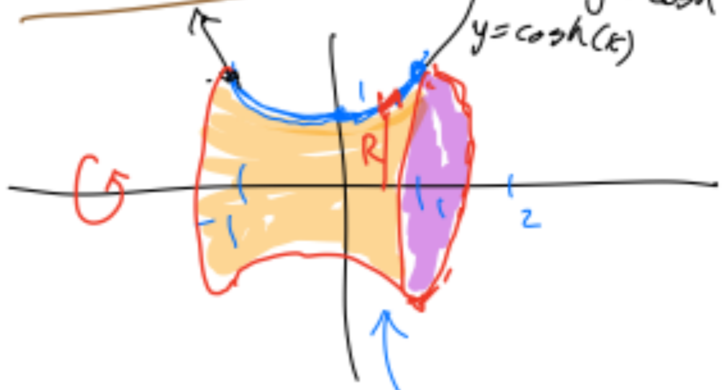
$$= \frac{\cancel{e^{2t}}+\cancel{e^{-2t}}+2-(\cancel{e^{2t}}+\cancel{e^{-2t}}-2)}{4}$$

$$= \frac{4}{4} = \boxed{1}.$$

$$[\sinh(t)]' = \left(\frac{e^t-e^{-t}}{2}\right)' = \frac{e^t+e^{-t}}{2} = \cosh(t)$$

$$[\cosh(t)]' = \left(\frac{e^t+e^{-t}}{2}\right)' = \frac{e^t-e^{-t}}{2} = \sinh(t).$$

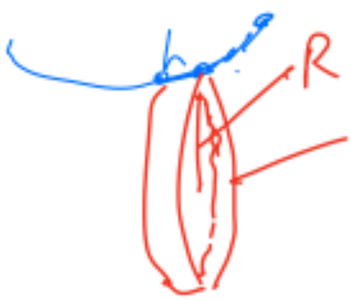
Example Find the surface area of the surface formed by revolving $y=\cosh(x)$ between $x=-1$ & $x=1$ around the x-axis.



$$y=\cosh(x)=\frac{e^x+e^{-x}}{2}$$

$$\cosh(1)=\frac{e^1+e^{-1}}{2}=\frac{e+\frac{1}{e}}{2}=\frac{e^2+1}{2e}$$

Called a "catenoid" ← example of a "minimal surface"



ribbon has surface area

$2\pi RL$

$\approx 2\pi R\sqrt{\Delta x^2+\Delta y^2}$

$$\Rightarrow \text{Total Surface Area} = \int_{x=-1}^{1} 2\pi R\sqrt{dx^2+dy^2}$$

$$=\int_{-1}^{1} 2\pi R\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$R=y=\cosh(x)$

$\frac{dy}{dx}=\sinh(x)$

$\Rightarrow$ Surface Area $= \displaystyle\int_{-1}^{1} 2\pi(\cosh(x))\sqrt{1+\sinh^2(x)}\,dx$

$\cosh^2 - \sinh^2 = 1$

$\cosh^2(x) = 1 + \sinh^2(x)$

$$= \int_{-1}^{1} 2\pi \underbrace{(\cosh(x))\cdot(\cosh(x))} \,dx$$

$$\left(\frac{e^x+e^{-x}}{2}\right)^2 = \frac{e^{2x}+e^{-2x}+2}{4}$$

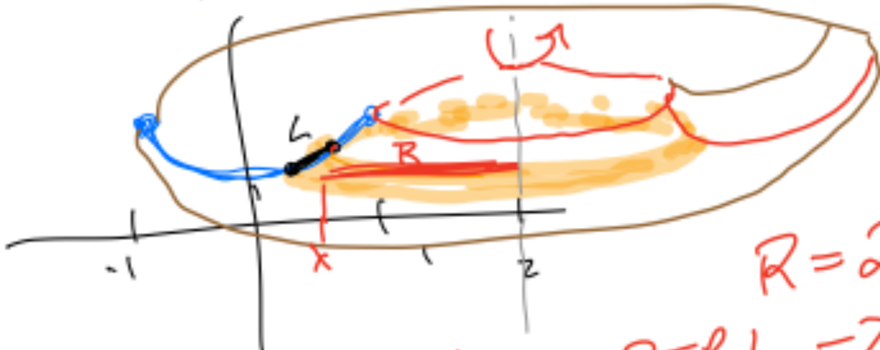
$$= \frac{\pi}{2}\int_{-1}^{1}\left(e^{2x}+e^{-2x}+2\right)dx$$

$$= \frac{\pi}{2}\left(\frac{e^{2x}}{2}+\frac{e^{-2x}}{-2}+2x\right)\Bigg|_{-1}^{1}$$

$$= \frac{\pi}{2}\left(\left(\frac{e^2}{2}-\frac{e^{-2}}{2}+2\right)-\left(\frac{e^{-2}}{2}-\frac{e^2}{2}-2\right)\right)$$

$$= \boxed{\frac{\pi}{2}\left(e^2-e^{-2}+4\right)}$$

$\boxed{\text{Example}}$ Find the surface area generated by rotating the curve $y = \cosh(x)$ between $x=-1$ & $x=1$ around the line $x=2$.



$R = 2-x$

Surface of slice $\approx 2\pi R L = 2\pi R\sqrt{\Delta x^2 + \Delta y^2}$

Total Surface Area $= \displaystyle\int_{x=-1}^{1} 2\pi R\sqrt{dx^2+dy^2}$

$$= \int_{x=-1}^{1} 2\pi(2-x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cosh(x)\right) = \sinh(x)$$

$$\sqrt{1+\sinh^2(x)} = \sqrt{\cosh^2(x)} = \cosh(x).$$

Surf Area $= \displaystyle\int_{-1}^{1} 2\pi(2-x)\cosh(x)\,dx = 2\pi\left((2-x)\sinh(x) + \int \sinh(x)\,dx\right)\Big|_{-1}^{1}$

parts
$u = 2-x \quad du = -dx$
$dv = \cosh(x)dx \quad v = \sinh(x)$

$$= \left((2-x)\sinh(x) + \cosh(x) \Big|_{-1}^{1} \right) 2\pi$$

$$= \left((2-1)\sinh(1) + \cosh(1) - \left((2-(-1))\sinh(-1) + \cosh(-1) \right) \right) 2\pi$$

$$\cosh(1) = \frac{e^{1}+e^{-1}}{2} = \frac{e+\frac{1}{e}}{2} = \frac{e^2+1}{2e}$$

$$\sinh(1) = \frac{e^{1}-e^{-1}}{2} = \frac{e-\frac{1}{e}}{2} = \frac{e^2-1}{2e}.$$

$$\cosh(-1) = \cosh(1) = \frac{e^2+1}{2e}$$

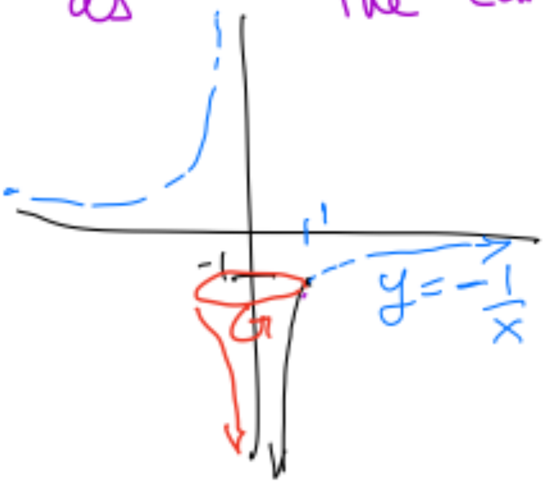
$$\sinh(-1) = -\sinh(1) = \frac{1-e^2}{2e}$$

$$= \left(\frac{e^2-1}{2e} + \frac{e^2+1}{2e} - \left[3\left(\frac{1-e^2}{2e}\right) + \frac{e^2+1}{2e} \right] \right) 2\pi$$

$$= \left(\frac{e^2-1+e^2+1-3+3e^2-e^2-1}{\cancel{2}e} \right) \cancel{2}\pi$$

$$= \left(\frac{4e^2-4}{e} \right)\pi = \boxed{4\pi\frac{(e^2-1)}{e}}$$

__Example__ Consider the infinite funnel, defined as the curve $y = \frac{-1}{x}$ for $0 < x \leq 1$, rotated around the y-axis.



(a) Find the surface area of the infinite funnel.

(b) Find the volume inside the infinite funnel.

(b)

$R = x$

$$\text{Volume} = \int_{y=-\infty}^{-1} \pi R^2 \, dy = \int_{y=-\infty}^{-1} \pi x^2 \, dy$$

$$y = \frac{-1}{x} \Rightarrow xy = -1$$

$$\Rightarrow x = \frac{-1}{y}.$$

$$\Rightarrow x^2 = \frac{1}{y^2}$$

$$\text{Volume} = \int_{-\infty}^{-1} \pi \frac{1}{y^2} \, dy = \lim_{a \to -\infty} \int_a^{-1} \pi \frac{1}{y^2} \, dy$$

$$= \lim_{a\to-\infty}\left(\pi \frac{-1}{y}\right)\Big|_a^{-1} = \lim_{a\to-\infty}\left(\pi\frac{(-1)}{(-1)} + \underset{\downarrow\, 0}{\pi\frac{1}{a}}\right)$$

$$= \boxed{\pi} \text{ finite!}$$

ⓐ Surface Area

$$\text{Surface Area} = \int 2\pi R L$$

$$R = x = \frac{-1}{y}$$

$$= \int 2\pi R\sqrt{dx^2+dy^2}$$

$$-y^{-1}$$

$$= \int_{-\infty}^{-1} 2\pi R\sqrt{\left(\frac{dx}{dy}\right)^2+1}\;dy$$

$$\frac{dx}{dy} = \frac{1}{y^2}$$

$$= \int_{y=-\infty}^{-1} 2\pi\left(\frac{-1}{y}\right)\sqrt{\frac{1}{y^4}+1\cdot\frac{y^4}{y^4}}\;dy$$

$$= -2\pi\int_{-\infty}^{-1}\frac{1}{y}\sqrt{\frac{y^4+1}{y^4}}\;dy$$

$$= +2\pi\int_{-\infty}^{-1}\sqrt{\frac{y^4+1}{y^6}}\;dy$$

as $y \to -\infty$, $\sqrt{\frac{y^4+1}{y^6}} \cong \sqrt{\frac{1}{y^2}} = \frac{1}{|y|}$

$= -\frac{1}{y}$

$$\frac{\sqrt{2}y^2}{y^3} = \frac{\sqrt{y^4+y^4}}{\sqrt{y^6}} \leq \sqrt{\frac{y^4+1}{y^6}}$$

$= \frac{\sqrt{2}}{y}$.

$$\frac{\sqrt{2}}{y} \leq \sqrt{\frac{y^4+1}{y^6}}$$

$$\underbrace{\int_{-\infty}^{-1} \frac{\sqrt{2}}{y}\,dy}_{=\infty} < \int_{-\infty}^{-1} \sqrt{\frac{y^4+1}{y^6}}\,dy$$

$\therefore$ $\boxed{\text{Surface area} = \boxed{\infty}}$

Alternate calculation:

$$\text{Surface Area} = \int 2\pi R L$$

$y = \frac{-1}{x}$ $\frac{dy}{dx} = \frac{1}{x^2}$

$$= \int 2\pi \overset{x}{\overset{=}{R}} \sqrt{dx^2+dy^2} = \int 2\pi x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^1 2\pi x \sqrt{1+\frac{1}{x^4}}\,dx$$

$$= \int_0^1 2\pi x \sqrt{\frac{x^4+1}{x^4}}\, dx$$

$$= \int_0^1 2\pi \frac{1}{x} \sqrt{1+\frac{1}{x^4}}\, dx$$

$$> \int_0^1 2\pi \frac{1}{x}\, dx = \infty.$$

Surface Area $= \infty$.

FREAKY Result: The infinite funnel has finite volume inside but ∞ surface area.

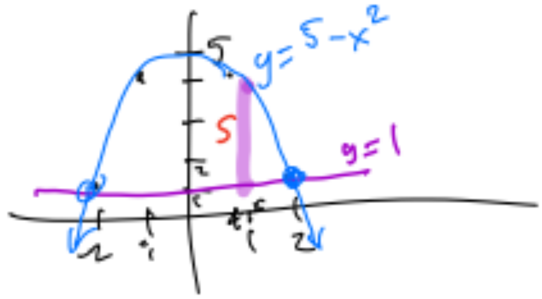
Question: This means we would not be able to paint the inside of the funnel with a finite amount of paint. However, we could fill the entire funnel with paint.

? ? ? ? ? ? ? ?

? Is this a contradiction? ? or a paradox??

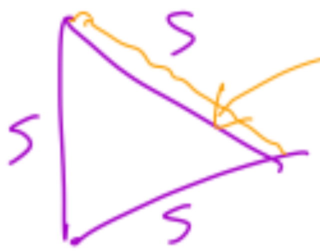
? ? ? ? ? ? ? ?

Example The base of a solid is the area between $y = 5 - x^2$ and $y = 1$. Cross sections of the solid perpendicular to the x-axis are equilateral triangles. Find the volume of the solid.



$$5 - x^2 = 1$$
$$4 = x^2$$
$$\pm 2 = x$$

Slices: thickened triangles



Thickness $= \Delta x$

$$\text{Volume of slice} = \begin{pmatrix}\text{Area of}\\ \text{triangle}\end{pmatrix} \Delta x$$

Area

$$= \frac{1}{2} b \cdot h = \frac{1}{2}(s)\left(\frac{\sqrt{3}}{2} s\right)$$

$$\text{Area} = \frac{\sqrt{3}}{4} s^2$$

$$\text{Total Volume} = \int (\text{volume of slice})$$

$$= \int_{-2}^{2} \left(\frac{\sqrt{3}}{4} s^2\right) dx$$

$$s = y_{top} - y_{bottom}$$
$$= (5 - x^2) - 1$$
$$s = 4 - x^2$$

$$\text{Total Volume} = \int_{-2}^{2} \frac{\sqrt{3}}{4} (4 - x^2)^2 \, dx$$

$$= \frac{\sqrt{3}}{4} \int_{-2}^{2} (16 - 8x^2 + x^4) \, dx$$

$$= \frac{\sqrt{3}}{4} \left(16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right)\Bigg|_{-2}^{2}$$

$$= \frac{\sqrt{3}}{4} \left(\left(16 \cdot 2 - \frac{8}{3} \cdot 8 + \frac{1}{5} 2^5\right) - \left(-32 + \frac{64}{3} - \frac{32}{5}\right) \right)$$

$$= \frac{\sqrt{3}}{4}\left(64 - \frac{128}{3} + \frac{64}{5}\right)$$

$$= \frac{64\sqrt{3}}{4}\left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$\frac{15 - 10 + 3}{15}$$

$$= 16\sqrt{3} \cdot \left(\frac{8}{15}\right) = \boxed{\frac{128\sqrt{3}}{15}}$$

Next application of integrals — in Physics — Calculating Work

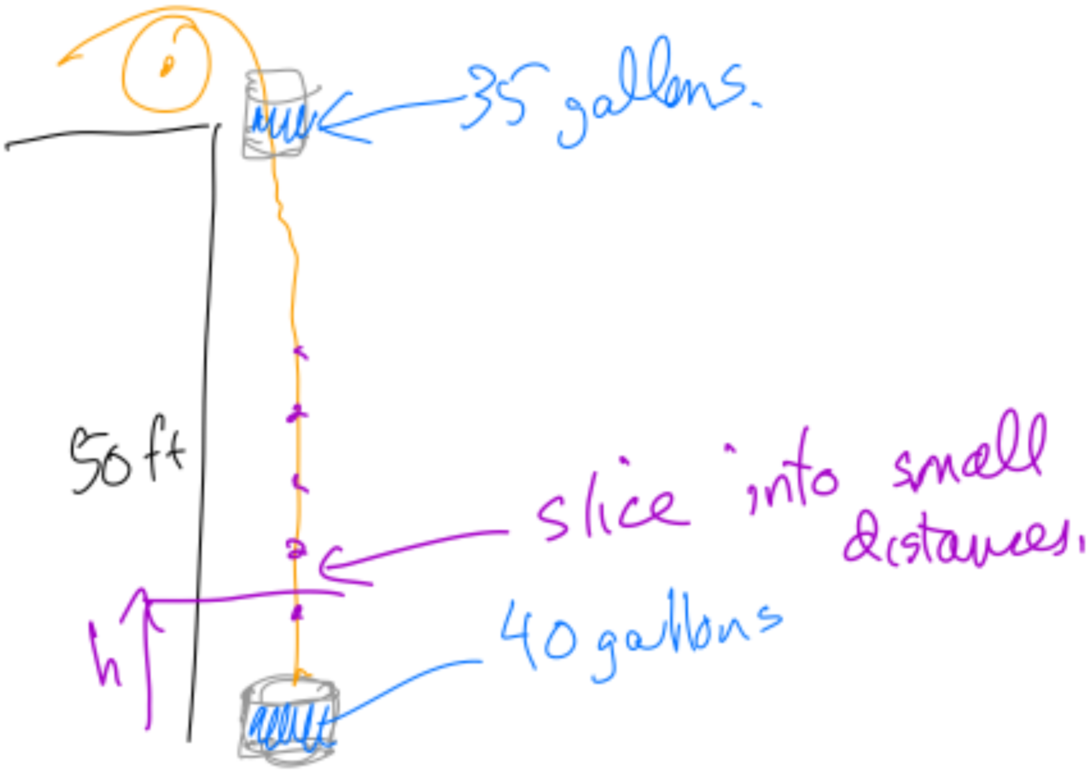
Work = Force · Distance
(if both are constant)
= energy required to do that force over that distance.

Example: When we pick up a 4 pound weight and lift it 3 feet,
Work required = (Force)(Distance)
= (4 pounds)(3 feet)
= 12 ft-pounds
= 3,889 calories.

If the force or distance are <u>not constant</u>:

$$\text{Work} = \int (\text{Force})(\text{Distance})$$

<u>Example</u>: A bucket that holds 40 gallons of water has a hole that leaks out 1 gallon of water per minute. You want to take this bucket with a rope to the top of a building that is 50 ft tall, and you are able to pull the rope continuously and get the bucket to the top in 5 minutes. How much work is required?



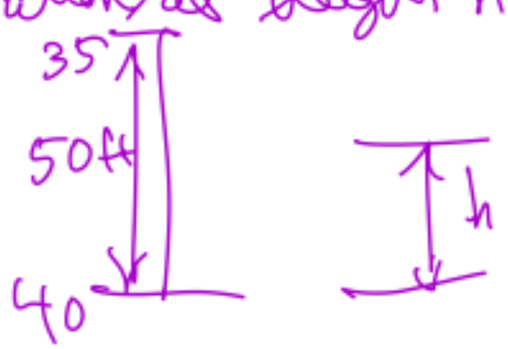
how much work = (force)(distance)

Δh

Force = weight of water

1 gallon = 8.34 pounds

gallons at height h



lose

$\frac{5 \text{ gallons}}{50 \text{ ft.}} = \frac{1 \text{ gallon}}{10 \text{ ft.}}$

$\frac{h}{10}$ gallons lost.

(gallons at height h) $= 40 - \frac{h}{10}$

(pounds at height h) $= \left(40 - \frac{h}{10}\right)(8.34)$.

Total Work $= \int (\text{Force})(\text{distance})$

$$= \int_0^{50} \left(40 - \frac{h}{10}\right)(8.34)_{\text{pounds}} \cdot dh$$

$$W = \left(40 \cdot 8.34\, h\right) - \frac{h^2}{20}(8.34) \Big|_0^{50}$$

$$= 40 \cdot (8.34)(50) - \frac{(50)^2}{20}(8.34)$$

$$= \boxed{15{,}637.5 \text{ ft pounds}}$$